

4.2. Duhamel's Method.

Consider nonhomogeneous heat conduction problem:

$$\boxed{\begin{aligned} \nabla^2 T(\vec{r}, t) + \frac{1}{k} g(\vec{r}, t) &= \frac{1}{\rho} \frac{\partial T(\vec{r}, t)}{\partial t} \\ \text{B.C. } K \frac{\partial T}{\partial n_i} \Big|_{S_i} + h_i T \Big|_{S_i} &= f_i(\vec{r}, t) \\ \text{I.C. } T(\vec{r}, t) \Big|_{t=0} &= F(\vec{r}) \end{aligned}}$$

Duhamel's method starts by considering a simpler auxiliary problem:

Let $\phi(\vec{r}, t, \tau)$ be the solution of the original problem on the assumption that the nonhomogeneous terms $g(\vec{r}, \tau)$ and $f_i(\vec{r}, \tau)$ do not depend on time t — the variable τ is merely a parameter but not a time variable.

$\phi(\vec{r}, t, \tau)$ is the solution of the following auxiliary problem:

$$\boxed{\begin{aligned} \nabla^2 \phi(\vec{r}, t, \tau) + \frac{1}{k} g(\vec{r}, \tau) &= \frac{1}{\rho} \frac{\partial \phi(\vec{r}, t, \tau)}{\partial t} \\ \text{B.C. } K \frac{\partial \phi(\vec{r}, t, \tau)}{\partial n_i} \Big|_{S_i} + h_i \phi(\vec{r}, t, \tau) \Big|_{S_i} &= f_i(\vec{r}, \tau) \\ \text{I.C. } \phi(\vec{r}, t, \tau) \Big|_{t=0} &= F(\vec{r}) \end{aligned}}$$

The auxiliary problem can be solved using separation of variable method, because τ is a parameter, and $g(\vec{r}, \tau)$ and $f_i(\vec{r}, \tau)$ do not depend on time t .

Assume $\phi(\vec{r}, t, \tau)$ is known, Duhamel's theorem relates the solution $T(\vec{r}, t)$ of the original problem to the solution $\phi(\vec{r}, t, \tau)$ of the auxiliary problem by the following expression:

$$T(\vec{r}, t) = \frac{\partial}{\partial t} \int_{\tau=0}^t \phi(\vec{r}, t-\tau, \tau) d\tau$$

equivalently: $T(\vec{r}, t) = \underbrace{F(\vec{r})}_{\text{Initial condition}} + \int_{\tau=0}^t \frac{\partial}{\partial t} \phi(\vec{r}, t-\tau, \tau) d\tau$.

4.3. Integral Method (approximate analytic method)

- * The results of integral method are approximate, but acceptable for many engineering applications.
- * Approximate analytic solutions provide an alternative to handling complicated nonhomogeneous conduction problems.
- * The Basic Steps. (for 1D case)

① Define a thermal layer ($\delta(t)$), there is no heat flow beyond $\delta(t)$ — so $\frac{\partial T}{\partial x} \Big|_{x=\delta} = 0$ (approximate), and the initial temperature is unaffected beyond $\delta(t)$ — so $T \Big|_{x=\delta(0)} = T_i \Big|_{x=\delta(0)}$ (approximate).

② The original conduction equation is integrated over $\delta(t)$. — $\int_{x=0}^{\delta(t)} dx$ — "energy integral equation"

③ choose an approximate temperature distribution for the thermal layer $0 < x < \delta(t)$. — A polynomial profile is preferred. The coefficients of the profile are determined in terms of $\delta(t)$ by using boundary conditions.

④ Introduce the approximate temperature profile into the "energy integral equation", obtain an ODE for $\delta(t)$, which can be solved using initial condition.

* Example

A semi-infinite medium ($x \geq 0$) is initially at T_i .

For times $t > 0$ the boundary surface is kept at T_0 .

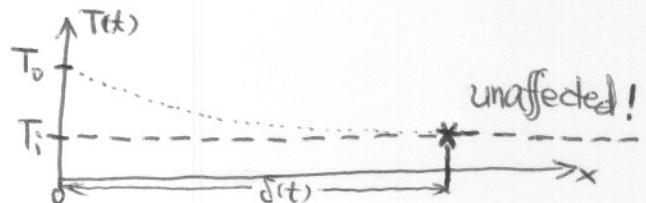
Determine the temperature using integral methods.

The complete problem:

$$\left. \begin{array}{l} \frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T(x,t)}{\partial t} \\ \text{B.C. } \left. T \right|_{x=0} = T_0 \\ \left. T \right|_{x \rightarrow \infty} = \text{finite} \\ \text{I.C. } \left. T \right|_{t=0} = T_i \end{array} \right\}$$

① Define a thermal layer $\delta(t)$ such that:

$$\left. \begin{array}{l} T \Big|_{x=\delta(t)} = T_i \\ \frac{\partial T}{\partial x} \Big|_{x=\delta(t)} = 0 \end{array} \right.$$



② Integrate original equation, from $0 \rightarrow \delta(t)$:

$$\int_{x=0}^{\delta(t)} \frac{\partial^2 T(x,t)}{\partial x^2} dx = \int_{x=0}^{\delta(t)} \frac{1}{\alpha} \frac{\partial T(x,t)}{\partial t} dx$$

$$\cancel{\left. \frac{\partial T}{\partial x} \right|_{x=\delta(t)}} - \left. \frac{\partial T}{\partial x} \right|_{x=0} = \frac{1}{\alpha} \int_{x=0}^{\delta(t)} \frac{\partial T(x,t)}{\partial t} dx$$

$$0 - \left. \frac{\partial T}{\partial x} \right|_{x=0} = \frac{1}{\alpha} \left[\frac{d}{dt} \int_{x=0}^{\delta(t)} T dx - \left. T \right|_{x=\delta(t)} \frac{d\delta}{dt} \right]$$

$$-\left. \frac{\partial T}{\partial x} \right|_{x=0} = \frac{1}{\alpha} \left[\int_{x=0}^{\delta(t)} T dx - T_i \frac{d\delta}{dt} \right]$$

Energy
integral
equation

③ choose an approximate temperature distribution:

$$T(x, t) = a(t) + b(t)x + c(t)x^2 + d(t)x^3 \quad [0 \leq x \leq \delta(t)]$$

For conditions are needed to determine coefficients:

$$\left\{ \begin{array}{l} T|_{x=0} = T_0 \quad (\text{original B.C.}) \\ T|_{x=\delta(t)} = T_i \quad (\text{thermal layer condition}) \\ \frac{\partial T}{\partial x}|_{x=\delta(t)} = 0 \quad (\text{thermal layer condition}) \\ \frac{\partial^2 T}{\partial x^2}|_{x=0} = 0 \quad (\text{because } T=T_0 = \text{const. at } x=0) \end{array} \right.$$

$\frac{\partial^2 T}{\partial x^2}|_{x=0} = \frac{1}{2} \frac{\partial T}{\partial t}|_{x=0}$

So: $\underbrace{\frac{T(x, t) - T_i}{T_0 - T_i}}_{=} = 1 - \frac{3}{2} \frac{x}{\delta(t)} + \frac{1}{2} \left(\frac{x}{\delta(t)} \right)^3$

④ Introduce $T(x, t)$ into "energy integral equation".

We have:

$$4\alpha = \underbrace{\int_0^{\delta(t)} \frac{d\delta(t)}{dt}}_{dt}$$

i.e. $\delta(t)|_{t=0} = 0$

so: $\underbrace{\delta(t) = \sqrt{8\alpha t}}$

Solution:

$$\underbrace{\frac{T(x, t) - T_i}{T_0 - T_i} = 1 - \frac{3}{2} \frac{x}{\sqrt{8\alpha t}} + \frac{1}{2} \left(\frac{x}{\sqrt{8\alpha t}} \right)^3}_{}$$